NOTES ON C_0-REPRESENTATIONS AND THE HAAGERUP PROPERTY

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Abstract. For any locally compact, second countable group G, we show the existence and uniqueness up to quasi-equivalence of a separable unitary C_0-representation π_0 of G such that all coefficient functions of C_0-representations of G are coefficient functions of π_0. The present work, strongly influenced by [4] (which dealt exclusively with discrete groups), leads to new characterizations of the Haagerup property: G has that property if and only if the representation π_0 induces a $*$-isomorphisme of $C^*(G)$ onto $C^*_{\pi_0}(G)$. When G is discrete, we also relate the Haagerup property to relative strong mixing properties of the group von Neumann algebra $L(G)$ into finite von Neumann algebras.

1. An enveloping C_0-representation

Throughout this article, G denotes a locally compact, second countable group, even though some results would hold in more general cases. We associate to G a unitary representation π_0 acting on a separable Hilbert space H_0 which has the following properties:

- it is a C_0-*representation*: every coefficient function $s \mapsto \langle \pi_0(s)\xi|\eta\rangle$ associated with π_0 tends to 0 as $s \to \infty$;
- every coefficient function of a C_0-representation is a coefficient function of π_0;
- the representation π_0 is the unique C_0-representation, up to quasi-equivalence, which satisfies the above properties.

The key idea is to use G. Arsac's notion of A_π-*spaces* from [1].

Using the same arguments as in Theorem 3.2 and Corollary 3.4 of [4], we deduce that:

Proposition A. *Let G be a group as above. Then it has the Haagerup property if and only if the maximal C^*-algebra $C^*(G)$ is $*$-isomorphic to the C^*-algebra $C^*_{\pi_0}(G)$.*

In the last part of the present notes, we assume that G is discrete. We relate the Haagerup property of G to the embedding of its von Neumann algebra $L(G)$ as a *strongly mixing* subalgebra of some finite von Neumann algebra M in the sense of [9]: based upon Chapter 2 of [5], we prove the following result (Theorem 2.5):

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Theorem B. *Let G be an infinite, countable group. Then it has the Haagerup property if and only if $L(G)$ can be embedded into some finite von Neumann algebra M in such a way that $L(G)$ is strongly mixing in M and that there is a sequence of elements $(x_k)_{k\geq 1}\subset M\ominus L(G)$ such that $\|x_k\|_2=1$ for every k, and*

$$\lim_{k\to\infty}\|\lambda(g)x_k-x_k\lambda(g)\|_2=0$$

for every $g\in G$.

In order to give precise statements of our results, we need to recall some notations and facts on spaces of coefficient functions of unitary representations (A_π-spaces of G. Arsac) from [1] and from P. Eymard's article [7].

The Banach algebra of all continuous functions on G which tend to 0 at infinity is denoted by $C_0(G)$, and its dense subalgebra formed by all continuous functions with compact support is denoted by $K(G)$.

Let (π,H) be a unitary representation of G. If $\xi,\eta\in H$, we denote by

$$\xi *_\pi \bar\eta(s)=\langle\pi(s)\xi|\eta\rangle\quad(s\in G)$$

the *coefficient function* associated to ξ and η. These functions are denoted by $\xi *_\pi \eta$ in [1] for instance, but our notation reminds the fact that $\xi *_\pi \bar\eta$ is linear in ξ and antilinear in η.

A representation (π,H) of G is a *C_0-representation* if, for all $\xi,\eta\in H$, the associated coefficient function $\xi *_\pi \bar\eta$ belongs to $C_0(G)$.

The *Fourier-Stieltjes algebra* is the set of all coefficient functions as above. It is denoted by $B(G)$ ([7]).

Recall that $B(G)$ is a Banach algebra with respect to the norm

$$\|\varphi\|_B=\inf\{\|\xi\|\|\eta\|:\varphi=\xi *_\pi \bar\eta\}.$$

It is the dual space of the enveloping C^*-algebra $C^*(G)$ under the duality bracket defined on the dense $*$-subalgebra $K(G)$

$$\langle\varphi,f\rangle=\int_G\varphi(s)f(s)ds\quad\forall\varphi\in B(G),\ f\in K(G).$$

Every unitary representation (π,H) of G gives rise to a natural $*$-homomorphism, still denoted by π, from $C^*(G)$ onto $C^*_\pi(G)$, which extends the map $f\mapsto\pi(f)$ defined on $K(G)$. (Recall that $C^*_\pi(G)$ is the C^*-algebra generated by $\{\pi(f):f\in K(G)\}$.)

If $E(G)$ is any subset of $B(G)$, we set

$$E_1(G)=\{\varphi\in E(G):\|\varphi\|_B=1\}$$

the intersection with the unit sphere of $B(G)$.

A continuous function $\varphi:G\to\mathbb{C}$ is *positive definite* if, for all $s_1,\dots,s_n\in G$ and all $t_1,\dots,t_n\in\mathbb{C}$, one has

$$\sum_{i,j=1}^n\bar t_i t_j\varphi(s_i^{-1}s_j)\geq 0.$$

We denote by $P(G)$ the set of all positive definite functions on G. For instance, every coefficient function $\xi *_\pi \bar\xi$ is positive definite, and, conversely, for every $\varphi\in P(G)$,

there exists a unique (up to unitary equivalence) triple $(\pi_\varphi, H_\varphi, \xi_\varphi)$ where (π_φ, H_φ) is a unitary representation of G and ξ_φ is a cyclic vector for π_φ that satisfies

$$\varphi = \xi_\varphi *_{\pi_\varphi} \bar{\xi}_\varphi.$$

We recall that $\|\varphi\|_B = \varphi(1)$ for every positive definite function φ.

If $\varphi \in B(G)$, the *adjoint* φ^* of φ is defined by $\varphi^*(s) = \overline{\varphi(s^{-1})}$ for every $s \in G$. We say that φ is *selfadjoint* if $\varphi^* = \varphi$ and we denote by $B_{sa}(G)$ the real Banach algebra of all selfadjoint elements of $B(G)$. Every element $\varphi \in B_{sa}(G)$ admits a unique decomposition, called *Jordan decomposition*, as

$$\varphi = \varphi^+ - \varphi^-$$

where $\varphi^\pm \in P(G)$ and $\|\varphi\|_B = \|\varphi^+\|_B + \|\varphi^-\|_B$. Thus $B_{sa}(G) = P(G) - P(G)$.

The obvious decomposition of any $\psi \in B(G)$

$$\psi = \frac{1}{2}(\psi + \psi^*) + i \cdot \frac{1}{2i}(\psi - \psi^*)$$

and the Jordan decomposition imply that

$$B(G) = P(G) - P(G) + iP(G) - iP(G).$$

We also need to recall the definition and a few facts on A_π-spaces in the sense of G. Arsac [1] since they play an important role in the present notes. If (π, H) is a unitary representation of G, $A_\pi(G)$ is the closed subspace of $B(G)$ generated by the coefficient functions $\xi *_\pi \bar{\eta}$ of π. Every element $\varphi \in A_\pi(G)$ can be written as

$$\varphi = \sum_n \xi_n *_\pi \bar{\eta}_n$$

where $\xi_n, \eta_n \in H$ for every n, $\sum_n \|\xi_n\|\|\eta_n\| < \infty$, and where

$$\|\varphi\|_B = \inf\{\sum_n \|\xi_n\|\|\eta_n\| : \varphi = \sum_n \xi_n *_\pi \bar{\eta}_n\}.$$

The Banach space $A_\pi(G)$ identifies with the predual of the von Neumann algebra $L_\pi(G) := \pi(G)'' \subset B(H)$ under the duality bracket

$$\langle \varphi, \pi(f) \rangle = \int_G \varphi(g) f(g) dg$$

for every $\varphi \in A_\pi(G)$ and every $f \in K(G)$.

As is usually the case, λ denotes the left regular representation of G, and $L(G) = L_\lambda(G)$ is its *associated von Neumann algebra*. In this case, $A(G) = A_\lambda(G)$ is the *Fourier algebra* of G ([7], Chapter 3).

If M is a von Neumann algebra, its predual is denoted by M_*, and if $\varphi \in M_*$ and $a \in M$, we define $a\varphi$ and $\varphi a \in M_*$ by

$$\langle a\varphi, x \rangle = \langle \varphi, xa \rangle \quad \text{and} \quad \langle \varphi a, x \rangle = \langle \varphi, ax \rangle \quad \forall x \in M.$$

Hence, one has $(a_1a_2)\varphi = a_1(a_2\varphi)$ and $\varphi(a_1a_2) = (\varphi a_1)a_2$ for all $\varphi \in M_*$ and $a_1, a_2 \in M$. If (π, H) is a unitary representation of G, if $\varphi = \sum_n \xi_n *_\pi \bar{\eta}_n \in A_\pi(G)$, then

$$\langle \varphi, x \rangle = \sum_n \langle x\xi_n | \eta_n \rangle \quad \forall x \in L_\pi(G).$$

If $a \in L_\pi(G)$, it is easily checked that

$$a\varphi = \sum_n (a\xi_n) *_\pi \bar{\eta}_n \quad \text{and} \quad \varphi a = \sum_n \xi_n *_\pi \overline{a^*\eta_n}.$$

Finally, if (π_1, H_1) and (π_2, H_2) are two unitary representations of G, then:

(1) we say that they are *quasi-equivalent* if the map $\pi_1(f) \mapsto \pi_2(f)$, from $\pi_1(K(G))$ to $\pi_2(K(G))$, extends to an isomorphism of $L_{\pi_1}(G)$ onto $L_{\pi_2}(G)$;
(2) we say that they are *disjoint* if no non-zero subrepresentation of π_1 is equivalent to some subrepresentation of π_2.

It follows from Propositions 3.1 and 3.12 of [1] that:

(a) the representations π_1 and π_2 are quasi-equivalent if and only if
$$A_{\pi_1}(G) = A_{\pi_2}(G);$$
(b) the representations π_1 and π_2 are disjoint if and only if
$$A_{\pi_1}(G) \cap A_{\pi_2}(G) = \{0\}.$$

Let us now introduce one of the main objects of the present article: let $A_0(G) = B(G) \cap C_0(G)$ be the space of all elements of $B(G)$ that tend to 0 at infinity. We also put $P_0(G) = P(G) \cap C_0(G)$, and let $A_{0,sa}(G)$ be the real subspace of selfadjoint elements of $A_0(G)$.

The following result is inspired by [4].

Proposition 1.1. *The set $A_0(G)$ is a closed two-sided ideal of $B(G)$, it is equal to the set of all coefficient functions of all C_0-representations and every $\varphi \in A_0(G)$ can be expressed as*

$$\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$$

with $\varphi_j \in P_0(G)$ for all $j = 1, \dots, 4$.

Proof. The space $A_0(G)$ is obviously a two-sided ideal of $B(G)$. It is closed because of the following inequality, which holds for every element $\varphi \in B(G)$:

$$\|\varphi\|_\infty \le \|\varphi\|_B.$$

Finally, the decomposition of φ as

$$\varphi = \frac{1}{2}(\varphi + \varphi^*) + i \cdot \frac{1}{2i}(\varphi - \varphi^*)$$

shows that it suffices to prove that for every selfadjoint element $\varphi \in A_0(G)$, the positive definite functions $\varphi^\pm$ of the Jordan decomposition $\varphi = \varphi^+ - \varphi^-$ both belong to $C_0(G)$. But it is proved in Lemme 2.12 of [7] that φ^+ and φ^- are uniform limits on G of linear combinations of right translates $s \mapsto \varphi(sg)$ of φ. As every such translate belongs to $C_0(G)$, this proves the claim. □

The reason why we denote the intersection $B(G) \cap C_0(G)$ by $A_0(G)$ instead of $B_0(G)$ for instance is that we will see that it is an A_π-space for some suitable representation that we introduce now.

The group G being second countable, we choose some dense sequence $(\varphi_n)_{n\ge1}$ in $P_{0,1}(G)$ and, for every integer $n \ge 1$, let (π_n, H_n, ξ_n) be the associated cyclic representation. Put first $K_0 = \bigoplus_{n\ge1} H_n$ and $\sigma_0 = \bigoplus_{n\ge1} \pi_n$. For instance, if G

is assumed to be discrete, one can set $\varphi_1 = \delta_1$, so that $\pi_1 = \lambda$ is the left regular representation of G. Next, set

$$H_0 = \bigoplus_{m\geq 1} K_0^{\otimes m} \quad \text{and} \quad \pi_0 = \bigoplus_{m\geq 1} \sigma_0^{\otimes m}.$$

Notice that both σ_0 and π_0 are C_0-representations.

Proposition 1.2. *Let G be a locally compact, second countable group, and let (π_0, H_0) be the above representation. Then:*

1. *For every C_0-representation π of G, one has $A_\pi(G) \subset A_0(G)$.*
2. *One has $A_0(G) = A_{\pi_0}(G)$, and every coefficient function of any C_0-representation is a coefficient function associated to π_0.*
3. *The unitary representation π_0 is the unique C_0-representation such that $A_0(G) = A_{\pi_0}(G)$, up to quasi-equivalence.*

Proof. (1) Observe that every coefficient function φ of the C_0-representation π is a linear combination of four elements in $P_{0,1}(G)$, by the same argument as in the proof of Proposition 1.1. As $A_0(G)$ is closed, this proves the first assertion. In particular, $A_{\sigma_0}(G)$ and $A_{\pi_0}(G)$ are contained in $A_0(G)$.
(2) First, if $\varphi \in P_{0,1}(G)$, then it is a limit of a subsequence (φ_{n_k}) of (φ_n). This shows that $\varphi \in A_{\pi_0}(G)$, and Proposition 1.1 proves that $A_0(G) \subset A_{\sigma_0}(G) \subset A_{\pi_0}(G)$. Next, let $\varphi \in A_0(G)$. Let us prove that it is a coefficient function of π_0. As $A_{\sigma_0}(G) = A_0(G)$, there exist sequences of vectors $(\xi_n)_{n\geq 1}, (\eta_n)_{n\geq 1} \subset K_0$ such that

$$\sum_n \|\xi_n\|\|\eta_n\| < \infty$$

and

$$\varphi = \sum_n \xi_n *_{\sigma_0} \bar{\eta}_n.$$

Replacing ξ_n by $\sqrt{\frac{\|\eta_n\|}{\|\xi_n\|}}\xi_n$ and η_n by $\sqrt{\frac{\|\xi_n\|}{\|\eta_n\|}}\eta_n$, we assume that

$$\sum_n \|\xi_n\|^2 = \sum_n \|\eta_n\|^2 = \sum_n \|\xi_n\|\|\eta_n\| < \infty.$$

Put $\xi = \bigoplus_n \xi_n, \eta = \bigoplus_n \eta_n \in H_0$. Then $\varphi = \xi *_{\pi_0} \bar{\eta}$.
(3) follows immediately from (1) and (2). $\square$

Definition 1.3. The representation (π_0, H_0) is called the **enveloping C_0-representation** of G.

Remark 1.4. As is well known, the left regular representation of G is a C_0-representation. Hence the Fourier algebra $A(G)$ is contained in $A_0(G)$. In fact, one can have equality $A(G) = A_0(G)$ as well as strict inclusion $A(G) \subsetneq A_0(G)$. Indeed, on the one hand, I. Khalil proved in [10] that if G is the $ax+b$-group over $\mathbb{R}$, then $A(G) = A_0(G)$, and, on the other hand, A. Figà-Talamanca [8] proved that if G is unimodular and if its von Neumann algebra $L(G)$ is not atomic (e.g. it is the case whenever G is infinite and discrete), then $A(G) \subsetneq A_0(G)$.

The next proposition is strongly inspired by, and is a slight generalization of Theorem 3.2 of [4]. It will be used to give characterizations of the Haagerup property in terms of the enveloping C_0-representation.

Proposition 1.5. *Let G be locally compact, second countable group and let (π, H) be a unitary representation of G, and let us assume that the space $A_\pi(G)$ is an ideal of $B(G)$. Then $\pi : C^*(G) \to C^*_\pi(G)$ is a $*$-isomorphism if and only if there is a sequence $(\varphi_n)_{n\geq 1} \subset A_\pi(G) \cap P_1(G)$ such that $\varphi_n \to 1$ uniformly on compact subsets of G.*

Proof. Assume first that π is a $*$-isomorphism. We can suppose that $C^*_\pi(G)$ contains no non-zero compact operator. Let χ be the state on $C^*_\pi(G)$ which comes from the trivial character $f \mapsto \int_G f(s)ds$ on $K(G) \subset C^*(G)$. By Glimm's Lemma, there exists an orthonormal sequence $(\xi_n)_{n\geq 1} \subset H$ such that

$$\chi(x) = \lim_{n\to\infty} \langle x\xi_n|\xi_n\rangle$$

for every $x \in C^*_\pi(G)$. Put $\varphi_n = \xi_n *_\pi \bar{\xi}_n \in A_\pi(G) \cap P_1(G)$ for every n. Then one has for every $f \in K(G)$:

$$\lim_{n\to\infty} \int_G \varphi_n(t)f(t)dt = \lim_{n\to\infty} \langle \pi(f)\xi_n|\xi_n\rangle = \int_G f(t)dt.$$

Theorem 13.5.2 of [6] implies that $\varphi_n \to 1$ uniformly on compact subsets of G. Conversely, if there exists a sequence $(\varphi_n)_{n\geq 1} \subset A_\pi(G) \cap P_1(G)$ such that $\varphi_n \to 1$ uniformly on compact subsets of G, let $x \in \ker(\pi)$. We have to prove that $\langle \varphi, x^*x\rangle_{B,C^*} = 0$ for every state φ on $C^*(G)$. Observe first that, for every $\psi \in A_\pi(G)$ and every $y \in C^*(G)$, one has

$$\langle \psi, y\rangle_{B,C^*} = \langle \psi, \pi(y)\rangle_{A_\pi, C^*_\pi}.$$

Indeed, if we write $\psi = \sum_k \xi_k *_\pi \bar{\eta}_k$, and if $f \in K(G)$, we have

$$\langle \psi, f\rangle_{B,C^*} = \int_G \psi(s)f(s)ds = \sum_k \int_G \langle \pi(s)\xi_k|\eta_k\rangle f(s)ds = \langle \psi, \pi(f)\rangle_{A_\pi, C^*_\pi}$$

and the formula holds by density of $K(G)$ in $C^*(G)$.

Let us fix such a state $\varphi \in P_1(G)$ and set $\psi_n = \varphi\varphi_n \in A_\pi(G) \cap P_1(G)$ for every n. As ψ_n is a state on $L_\pi(G)$, its restriction to $C^*_\pi(G)$ is still a state, and $\langle \psi_n, x^*x\rangle = \langle \psi_n, \pi(x^*x)\rangle = 0$ for every n. As $\psi_n \to \varphi$ in the weak* topology of $B(G) = C^*(G)^*$, one has $\langle \varphi, x^*x\rangle = 0$. □

2. The Haagerup property

As in the first section, G denotes a locally compact, second countable group, and (π_0, H_0) denotes its enveloping C_0-representation.

Following M. Bekka [3], we say that (π, H) is an *amenable representation* if $\pi \otimes \bar{\pi}$ weakly contains the trivial representation. Equivalently, this means that there exists a sequence of unit vectors $(\xi_n)_{n\geq 1} \subset H \otimes \bar{H}$ such that

$$\langle \pi \otimes \bar{\pi}(s)\xi_n|\xi_n\rangle \to 1$$

uniformly on compact subsets of G; notice that $\pi \otimes \bar{\pi}$ is unitarily equivalent to the representation $(T, g) \mapsto \pi(g)T\pi(g^{-1})$ acting on the space $HS(H)$ of all Hilbert-Schmidt operators.

We say that G has the *Haagerup property* if there exists a sequence $(\varphi_n)_{n\geq 1} \subset P_{0,1}(G)$ which tends to 1 uniformly on compact sets. Note that it is equivalent to say that G admits an amenable, C_0-representation. See [5] for more information on the Haagerup property.

The next result generalizes partly, and is inspired by Corollary 3.4 of [4].

Proposition 2.1. *Let G and (π_0, H_0) be as above. Then the following conditions are equivalent:*

(1) *G has the Haagerup property;*
(2) *$C^*(G) = C^*_{\pi_0}(G)$, i.e. the $*$-homomorphism $\pi_0 : C^*(G) \to C^*_{\pi_0}(G)$ is an isomorphism;*
(3) *the representation π_0 weakly contains the trivial representation;*
(4) *the representation π_0 is amenable.*

Proof. (1) $\Rightarrow$ (2). There exists a sequence $(\varphi_n)_{n\geq 1} \subset P_{0,1}(G)$ which converges to 1 uniformly on compact sets. The assertion follows readily from Proposition 1.5.
(2) $\Rightarrow$ (3). It follows also from Proposition 1.5.
(3) $\Rightarrow$ (4) et (4) $\Rightarrow$ (1) are obvious. □

Remark 2.2. As $A(G) \subset A_{\pi_0}(G)$, there exists a $*$-homomorphism Φ from $L_{\pi_0}(G)$ onto $L(G)$ such that $\Phi(\pi_0(f)) = \lambda(f)$ for every $f \in K(G)$. Thus, let $z_A \in L_{\pi_0}(G)$ be the central projection such that $L_{\pi_0}(G)z_A$ is $*$-isomorphic to $L(G)$. This allows us to consider the following two subrepresentations of π_0: set $\pi_{00}(s) = \pi_0(s)(1-z_A)$ and $\lambda_0(s) = \pi_0(s)z_A$ for all $s \in G$. Then λ_0 is quasi-equivalent to λ, and since π_{00} is disjoint from λ, we have $A_{\pi_{00}}(G) \cap A(G) = \{0\}$. It would be interesting to get more information on π_{00}, in particular when G has the Haagerup property.

From now on, we assume that G is an infinite, discrete (countable) group. Following [4], for any (not necessarily closed) ideal $D \subset \ell^\infty(G)$, we say that a unitary representation (π, H) of G is a *D-representation* if H contains a dense subspace K such that the coefficient function $\xi *_\pi \bar{\eta} \in D$ for all $\xi, \eta \in D$. We associate to D the following C^*-algebra $C^*_D(G)$: it is the completion of $K(G)$ with respect to the C^*-norm

$$\|f\|_D := \sup\{\|\pi(f)\| : \pi \text{ is a } D-\text{representation}\}.$$

When $D = C_0(G)$, one gets $C^*_D(G) = C^*_{\pi_0}(G)$. This makes the link between Proposition 2.1 above and the main results of N. Brown and E. Guentner in [4].

We end the present notes with a relationship between the Haagerup property for discrete groups and strongly mixing von Neumann subalgebras in the sense of [9], Definition 1.1. We need to recall some definitions and facts from [9] first and from Chapter 2 of [5] next.

Let $1 \in B \subset M$ be finite von Neumann algebras (with separable preduals) endowed with a normal, finite, faithful, normalized trace τ. We denote by $\mathbb{E}_B$ the τ-preserving conditional expectation from M onto B, and by $M \ominus B = \{x \in M : \mathbb{E}_B(x) = 0\}$. We assume that B is diffuse.

Definition 2.3. Let $B \subset M$ be a pair as above. We say that B is **strongly mixing in** M if

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0$$

for all $x, y \in M \ominus B$ and all sequences $(u_n) \subset U(B)$ which converge to 0 in the weak operator topology.

This definition is motivated by the following situation: if a countable group G acts in a trace-preserving way on some finite von Neumann algebra (Q, τ) and if we put $B := L(G) \subset M := Q \rtimes G$, then B is strongly mixing in M if and only if

the action of G on Q is strongly mixing in the usual sense: for all $a, b \in Q$, one has $\lim_{g\to\infty} \tau(a\sigma_g(b)) = \tau(a)\tau(b)$.

Let now G be a countable group with the Haagerup property. By Theorems 2.1.5, 2.2.2 and 2.3.4 of [5], there exists a trace preserving and strongly mixing action of G on some finite von Neumann algebra (Q, τ) which contains non trivial asymptotically invariant sequences and Følner sequences in the sense below. For instance, if G has the Haagerup property, there exists an action α of G on the hyperfinite type II_1-factor R such that:

- α is strongly mixing;
- the fixed point algebra $(R_\omega)^\alpha$, that is, the set of all (classes of) central sequences $x = [(x_n)] \in R_\omega$ such that $\alpha_g^\omega(x) = x$ for all $g \in G$, is of type II_1.

Definition 2.4. Let $1 \in B \subset M$ be a pair of finite von Neumann algebras as above, and let $(e_k)_{k\geq 1} \subset M$ be a sequence of projections in M.

(1) We say that $(e_k)_{k\geq 1}$ is a **non trivial asymptotically invariant sequence** for B if $\mathbb{E}_B(e_k) = \tau(e_k)$ for every k, if
$$\lim_{k\to\infty} \|be_k - e_k b\|_2 = 0$$
for every $b \in B$ and if
$$\inf_k \tau(e_k)(1 - \tau(e_k)) > 0.$$

(2) We say that $(e_k)_{k\geq 1}$ is a **Følner sequence** for B if $\mathbb{E}_B(e_k) = \tau(e_k)$ for every k, if $\lim_k \|e_k\|_2 = 0$ and if
$$\lim_{k\to\infty} \frac{\|be_k - e_k b\|_2}{\|e_k\|_2} = 0$$
for every $b \in B$.

In general, the existence of a non trivial asymptotically invariant sequence for B implies the existence of a Følner sequence for B, but the converse does not hold. See [5], p. 19, for more details.

Combining these types of properties, we get:

Theorem 2.5. *Let G be an infinite, countable group. Then it has the Haagerup property if and only if it satifies one of the following equivalent conditions:*

(1) *(resp. (1′)) There exists a finite von Neumann algebra M containing $L(G)$ such that $L(G)$ is strongly mixing in M and M contains a Følner sequence for $L(G)$ (resp. a non trivial asymptotically invariant sequence for $L(G)$).*

(2) *There exists a finite von Neumann algebra M containing $L(G)$ such that $L(G)$ is strongly mixing in M and there is a sequence of elements $(x_k)_{k\geq 1} \subset M \ominus B$ such that $\|x_k\|_2 = 1$ for every k, and*
$$\lim_{k\to\infty} \|\lambda(g)x_k - x_k\lambda(g)\|_2 = 0$$
for every $g \in G$.

Proof. If G has the Haagerup property, then each condition (1), (1') and (2) holds, by Theorem 2.3.4 of [5], and that there are plenty of non trivial asymptotically invariant or Følner sequences in the hyperfinite type II_1-factor R. Thus, assume that condition (1) holds and that $B := L(G)$ embeds into some finite von Neumann algebra M such that $B := L(G)$ is strongly mixing in M and that M

contains a Følner sequence for B. We have to show the existence of a sequence $(\varphi_k)_{k\geq 1}\subset P_{0,1}(G)$ which tends to 1 pointwise.

Recall first that to any completely positive map $\Phi : M\to M$, one associates a function φ on G by

$$\varphi(g)=\tau(\Phi(\lambda(g))\lambda(g^{-1}))\quad (g\in G),$$

and that φ is positive definite. In particular, for every $x\in M\ominus B$, the function $\varphi_x : G\to\mathbb{C}$ defined by

$$\varphi_x(g)=\tau(\mathbb{E}_B(x^*\lambda(g)x)\lambda(g^{-1}))=\tau(x^*\lambda(g)x\lambda(g^{-1}))\quad (g\in G)$$

is positive definite. Moreover, since B is strongly mixing in M and since $\lambda(G)$ is an orthonormal set, one has

$$|\varphi_x(g)|\leq \|\mathbb{E}_B(x^*\lambda(g)x)\|_2\to 0$$

as $g\to\infty$, which shows that $\varphi_x\in P_0(G)$ for every $x\in M\otimes B$.
Next, let $(e_k)_{k\geq 1}\subset M$ be a Følner sequence for B and choose $c>0$ and an integer $k_0>0$ such that

$$1-\tau(e_k)\geq c$$

holds for every $k\geq k_0$. Define then

$$x_k=\frac{e_k-\tau(e_k)}{\sqrt{\tau(e_k)(1-\tau(e_k))}}(=x_k^*)\quad (k\geq 1)$$

and put $\varphi_k=\varphi_{x_k}$ for every k. One has, for every integer $k\geq k_0$ and every $g\in G$:

$$\begin{aligned}\varphi_k(g) &= \tau(x_k\lambda(g)x_k\lambda(g^{-1}))\\ &= \frac{1}{\tau(e_k)(1-\tau(e_k))}\cdot\tau((e_k-\tau(e_k))\lambda(g)(e_k-\tau(e_k))\lambda(g^{-1}))\\ &= \frac{1}{\tau(e_k)(1-\tau(e_k))}\cdot\tau(e_k\lambda(g)e_k\lambda(g^{-1})-\tau(e_k)^2)\\ &= \frac{\tau(e_k(\lambda(g)e_k\lambda(g^{-1})-e_k))}{\tau(e_k)(1-\tau(e_k))}+1.\end{aligned}$$

Hence, by Cauchy-Schwarz Inequality,

$$\begin{aligned}|\varphi_k(g)-1| &\leq \frac{1}{c}\cdot\frac{\|e_k\|_2\|\lambda(g)e_k\lambda(g^{-1})-e_k\|_2}{\|e_k\|_2^2}\\ &= \frac{1}{c}\cdot\frac{\|\lambda(g)e_k-e_k\lambda(g)\|_2}{\|e_k\|_2}\to 0\end{aligned}$$

as $k\to\infty$ for every $g\in G$. A similar argument works if (e_k) is a non trivial asympotically invariant sequence.

Finally, assume that G satisfies condition (2), and let $(x_k)\subset M\ominus B$ be as above. Define $\varphi_k(g)=\tau(x_k^*\lambda(g)x_k\lambda(g^{-1}))$ exactly as above. Then by the same arguments, $\varphi_k\in P_{0,1}(G)$ for every k, and, for fixed $g\in G$, one has:

$$\begin{aligned}|\varphi_k(g)-1| &= |\tau(x_k^*\lambda(g)x_k\lambda(g^{-1}))-\tau(x_k^*x_k)|\\ &= |\langle\lambda(g)x_k\lambda(g)-x_k|x_k\rangle|\\ &\leq \|\lambda(g)x_k\lambda(g^{-1})-x_k\|_2\|x_k\|_2\\ &= \|\lambda(g)x_k\lambda(g^{-1})-x_k\|_2\to 0\end{aligned}$$

as $k\to\infty$. $\square$

Remark 2.6. Assume that G has the Haagerup property. One can ask whether there exists a group Γ containing G and such that the pair of finite von Neumann algebras $L(G) \subset L(\Gamma)$ satisfies condition (2) in Theorem 2.5. Unfortunately, it is only the case when G is amenable, and this has no real interest. Indeed, assume for simplicity that G is torsion free, that it embeds into some group Γ and that the pair $L(G) \subset L(\Gamma)$ satisfies condition (2) above. Then, on the one hand, by Lemma 2.2 and Proposition 2.3 of [9], the pair of groups $G \subset \Gamma$ satisfies *condition (ST)*, which means that, for every $\gamma \in \Gamma \setminus G$, the subgroup $\gamma G \gamma^{-1} \cap G$ is finite, hence trivial. In other words, G is *malnormal* in Γ. On the other hand, by classical arguments, the existence of a sequence $(x_k) \subset L(\Gamma) \ominus L(G)$ as above implies that the action $G \curvearrowright X := \Gamma \setminus G$ defined by $(g, x) \mapsto gxg^{-1}$ has an invariant mean. This means that the associated representation λ_X weakly contains the trivial representation. But the first condition implies that this action is free, hence that λ_X is equivalent to a multiple of the regular representation. This forces G to be amenable.

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